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## ESTIMATION ERROR AND PORTFOLIO OPTIMIZATION: A RESAMPLING SOLUTION\*

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*Markowitz (1959) mean–variance (MV) portfolio optimization has been the practical standard for asset allocation and equity portfolio management for almost 50 years. However, it is known to be overly sensitive to estimation error in risk–return estimates and have poor out-of-sample performance characteristics. The Resampled Efficiency™ (RE) techniques presented in Michaud (1998) introduce Monte Carlo methods to properly represent investment information uncertainty in computing MV portfolio optimality and in defining trading and monitoring rules. This paper reviews and updates the literature on estimation error and RE portfolio optimization and rebalancing. We resolve several open issues and misunderstandings that have emerged since Michaud (1998). In particular, we show RE optimization to be a Bayesian-based generalization and enhancement of Markowitz’s solution.*



Markowitz (1959) mean–variance (MV) optimization has been the standard for efficient portfolio construction for almost 50 years. Nearly all commercial portfolio optimizers for asset allocation

and equity portfolio management are based on some variation of the Markowitz method. While theoretically important for modern finance, MV optimization’s sensitivity to uncertainty in risk–return estimates typically results in an unstable asset management framework, ambiguous portfolio optimality, and poor out-of-sample performance. Tests demonstrate that unbounded MV optimized portfolios are dominated by equal weighting and have essentially no practical investment value.<sup>1</sup>

In practice MV optimization is used primarily as a convenient framework for imposing *ad hoc*

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constraints and providing a scientific veneer for marketing purposes (Michaud, 1989). We show that MV optimization limitations result primarily from the way investment information is used. Resampled Efficiency™ (RE) optimization and rebalancing, first proposed in Michaud (1998, Chs. 6, 7), uses Monte Carlo techniques to define more investment effective Markowitz optimized portfolios and portfolio rebalancing and monitoring rules.<sup>2</sup>

This paper reviews and summarizes recent research and new developments in estimation error and MV portfolio optimization.<sup>3</sup> It includes resolutions of open issues and misunderstandings that emerged since Michaud (1998). The arguments are briefly summarized to keep the report of manageable length, with references provided for further discussion.

## 1 Research review

This paper addresses linear constrained MV portfolio optimization and rebalancing from a statistical or estimate uncertainty perspective with evaluation in an out-of-sample context. The pioneers of an estimate uncertainty statistical perspective of MV optimization include Roll (1979), Jobson and Korkie (1980, 1981), Shanken (1985), Jorion (1986, 1992), Frost and Savarino (1986, 1988), and Ledoit (1997). Curiously, the bulk of traditional research in this area has focused on unbounded in-sample MV optimization or utility maximization while ignoring estimate uncertainty or out-of-sample performance. Examples include the Black and Litterman (1992) procedure, Grinold (1989) formula, Grinold and Kahn (1994, Ch. 6) principles, Clarke *et al.* (2002, 2006) rationale, Campbell and Viceira (2002) utility, and Knight and Satchell (2006) analysis. While an

in-sample analytical framework may be tractable and convenient, unbounded MV optimization without consideration of estimation error often leads to irrelevant or misleading trading, optimization universe design, and constraint policies that may adversely affect trillions of dollars under current management.<sup>4</sup>

Academic research on estimation error has largely focused on (unbounded in-sample) expected utility maximization.<sup>5</sup> However, practitioners typically prefer MV efficiency for defining portfolio optimality since investors are more comfortable estimating asset risks and returns than parameters of a utility function. Levy and Markowitz (1979) show that portfolios on the MV efficient frontier are often very good approximations of portfolios that maximize expected utility for many utility functions and return generating processes in practice.<sup>6</sup> In addition, Rubenstein (1973) shows that parameter uncertainty is often a serious problem in expected utility estimation. While maximum expected utility using sophisticated functions or higher moments of the distribution may claim superiority over MV optimization in-sample, increased estimation error may often lead to inferior out-of-sample risk-adjusted performance.

## 2 Scope

Our focus—out-of-sample risk-adjusted performance of linearly constrained MV optimized portfolios—is the framework of choice for asset management in practice. Following Jobson and Korkie (1981), our out-of-sample results are based on simulation tests to avoid the unreliability of backtests and unrealistic in-sample analytical results. Computer algorithms that include linear constraints as in Markowitz (1956) are used to compute practical MV optimal portfolios.<sup>7</sup>

### 3 MV optimization limitations

The problem that limits the investment value of MV optimized portfolios is not Markowitz' theory.<sup>8</sup> Markowitz gives the right way to invest given that you know that you have exactly the correct inputs. The most serious problem is estimation error, or parameter uncertainty, in optimization inputs. Risk-return estimates are highly uncertain in investment practice and sensitivity to changes in optimization inputs leads to portfolio optimality ambiguity. The problem of uncertain estimates is compounded by how investment information is represented in digital computer optimizations. A 10% return estimate is stored in computer registers with 15 trailing zeros.<sup>9</sup> In practice it is investment nonsense to consider that significant information exists in the 16th or even 5th decimal place; (many investment professionals admit they are happy to just get the sign right). However, this extremely unrealistic view of investment information is implicit in the functioning of many optimization algorithms.<sup>10</sup> What is missing is any sense of statistically significant differences in risk-return inputs, a particular problem in the context of optimization with inequality constraints. As a result, MV optimization creates unrealistic portfolios leading to likely underperformance even with informative inputs.

### 4 RE technology

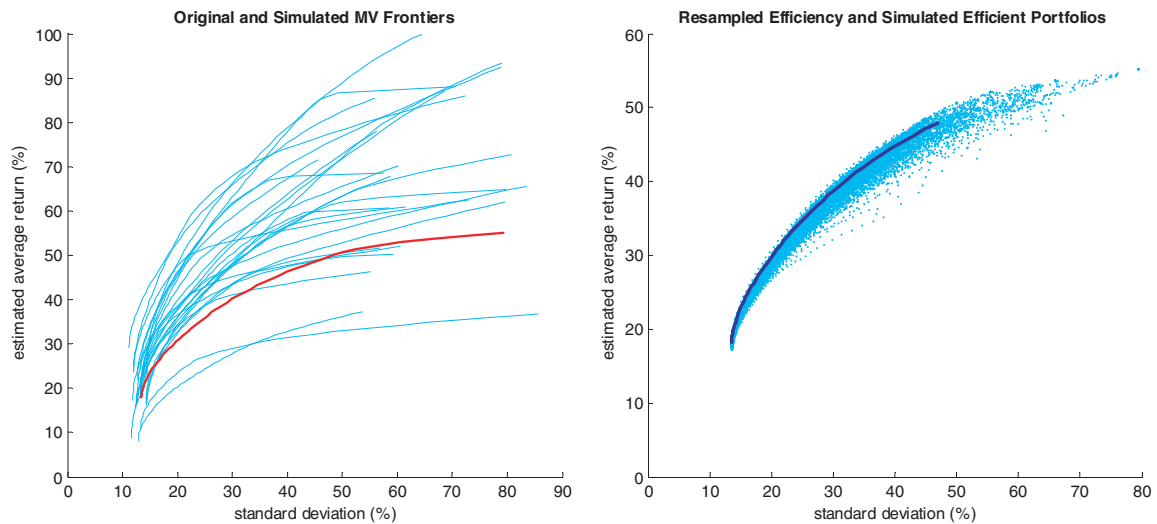
RE technology introduces Monte Carlo resampling and bootstrapping methods into MV optimization to more realistically reflect the uncertainty in investment information. The end result is generally more stable, realistic, and investment effective MV optimized portfolios. RE technology also includes statistically rigorous portfolio trading and monitoring rules and tests for significant assets avoiding the often ineffective and costly rebalancings

typical of the MV optimization asset management process.

Generally speaking, RE optimal portfolios are constructed as follows (each step is discussed in more depth later in the paper):

- Step 1. Sample a mean vector and covariance matrix of returns from a distribution of both centered at the original (point estimate) values normally used in MV optimization.<sup>11</sup>
- Step 2. Calculate a MV efficient frontier based on these sampled risk and return estimates.
- Step 3. Repeat Steps 1 and 2 (until enough observations are available for convergence in Step 4).
- Step 4. Average the portfolio weights from Step 2 to form the RE optimal portfolio.
- Step 5. (optional) Apply investability constraints to 4.

The left-hand panel in Figure 1 illustrates the resampled simulated MV efficient frontier procedure for 20 randomly chosen large cap stocks in the S&P 500 index.<sup>12,13</sup> The red curve is the classical Markowitz MV efficient frontier. Twenty-five (cyan) simulated MV efficient frontiers, using the repeated two steps, are also displayed.<sup>14</sup> Note the enormous dispersion of the simulated frontiers. Some simulated frontiers have roughly half the range of risk of the original MV efficient frontier while others have significantly more risk. The range of returns among the simulated frontiers is even more impressive considering that the experiment reflects 10 years of data from well known securities in a major index. Since each of these frontiers represents MV portfolio optimality, MV efficiency may seem almost beyond hope of definition when estimation error is considered. The left-hand panel provides a vivid rationale for why investment managers typically avoid using MV optimization in practice or over manage the process and impose many constraints.<sup>15</sup>



**Figure 1** Original and simulated MV frontiers and the Resampled Efficient Frontier™.

Each simulated efficient frontier in the left-hand panel consists of 51 portfolios from low to high return. The right-hand panel plots the risks and returns of all the portfolios computed from all the simulated efficient frontiers in the left-hand panel in terms of the Appendix risk-return estimates. RE optimization converts the simulated MV efficient frontiers into a single frontier, as displayed in the right-hand panel.

The procedure for computing the Resampled Efficient Frontier™ (REF) in the right-hand panel can be motivated as follows. Every simulated MV efficient frontier in the left-hand panel of Figure 1 is the right way to invest given a set of inputs. But the inputs are highly uncertain. How should an investor deal with portfolio optimality uncertainty?

In the case of a highly risk-averse investor, the minimum variance portfolio is the optimal portfolio for any given efficient frontier. Since all simulated frontiers are equally likely, the RE optimal minimum variance portfolio is defined as the average of the portfolio weights of all the simulated minimum variance portfolios. The RE optimal minimum variance portfolio is plotted at the base of the curve

in the right-hand panel in Figure 1 relative to the risk-return estimates of the data in the Appendix.

Alternatively, consider an investor who is indifferent to risk. In this case, the maximum return portfolio is the optimal portfolio for any given simulated efficient frontier. The RE optimal maximum return portfolio is defined as the average of the portfolio weights of all the simulated maximum return MV optimal portfolios; it is plotted at the top of the curve in the right-hand panel.

Similarly, RE optimality can be defined for the utility function that characterizes an investor's behavior. The average of the tangent portfolios on simulated MV efficient frontiers relative to the constant expected utility curve defines the RE optimal portfolio. The Resampled Efficient Frontier is the collection of all possible RE optimal portfolios with risk aversion parameters from expected utility curves ranging from total risk aversion to total risk indifference.<sup>16</sup> More generally, the REF is based on averages of all properly associated optimal portfolios on the simulated MV efficient frontiers. Mathematically, it is an integral in portfolio space, over all possible return distributions consistent with the

forecast, of the expected value of the MV optimal portfolio weights conditional on the constraints. The resampling/bootstrap process is a Monte Carlo method for estimating this integral.<sup>17</sup> Since REF portfolios are an average of many properly associated MV optimal portfolios, they are safer, less extreme investments.

### 5 Properties of MV vs. RE optimization

Figure 2 displays the RE and MV frontiers for the data in the Appendix. The REF plots below the classical efficient frontier. Superficially, RE optimization appears to be inferior as an investment framework because it expects less return and restricts risk to a narrower range.<sup>18</sup> Any in-sample study, as in Harvey *et al.* (2006), will conclude that the REF is not optimal since in-sample utility is unlikely to be maximized by the REF. Note that constraints do the same thing, i.e., lower and shorten the frontier. RE optimization can be thought of as a constraint based on the level of information in the forecasts. In practice, analysts nearly always impose constraints with the objective of improving

performance. Frost and Savarino (1988) show that constraints, while lowering in-sample expectations, improve out-of-sample performance.

The proper interpretation of REF vs. classical MV optimality is simple to explain. If you are 100% certain of your risk-return estimates (to 16 decimal places of accuracy or more) then the Markowitz efficient frontier is the efficient frontier for you. If you are less than 100% certain of your estimates of risk and return, you expect less return and are less willing to put money at risk. Furthermore, consider an investor with a complete lack of certainty in investment information. In this case the optimal efficient frontier is the no-information prior portfolio, either equal- or benchmark-weighted.<sup>19</sup> The REF portfolio collapses to the no-information portfolio, while Markowitz optimization remains insensitive to information uncertainty. RE optimization is the paradigm of choice for rational decision making under conditions of information uncertainty.

Figure 2 illustrates that RE and MV frontiers may be close in MV space. Superficially this may

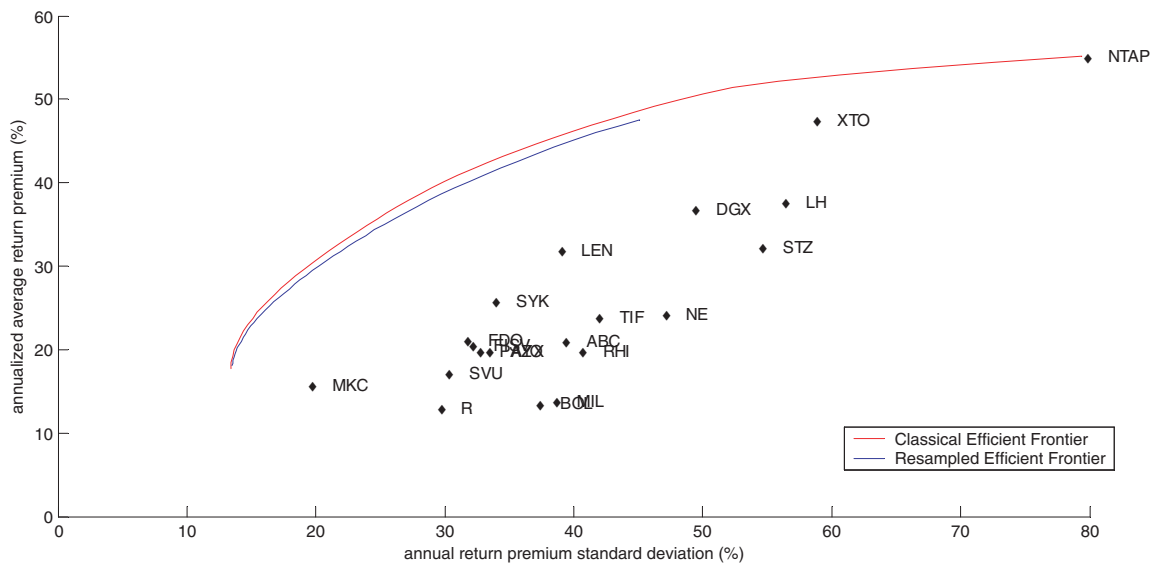
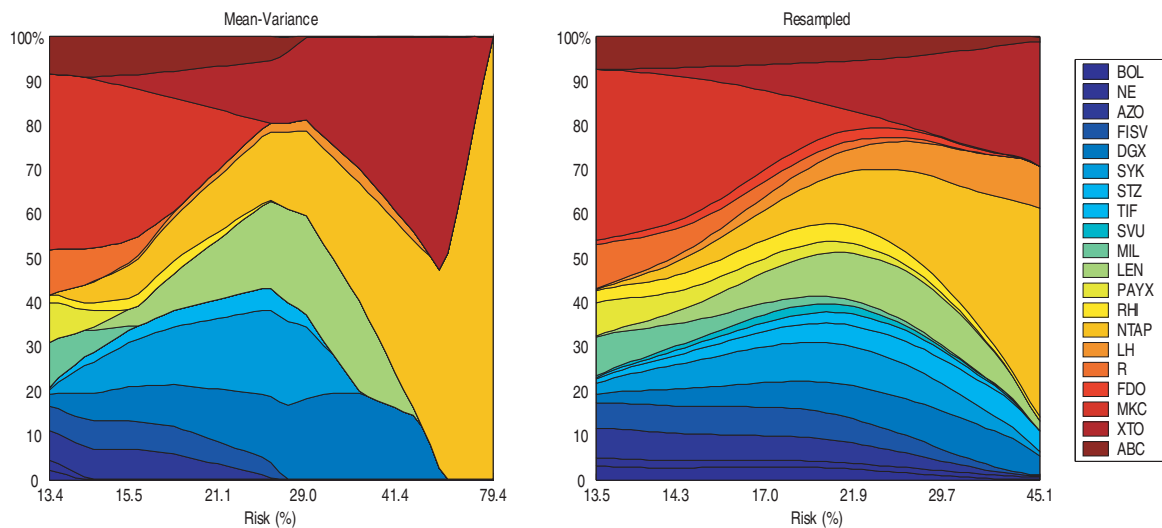


Figure 2 Classical and Resampled Efficient Frontiers.



**Figure 3** MV and RE frontier portfolio composition maps.

suggest that the procedures produce similar solutions. Figure 3 provides an illustration of portfolio composition differences in MV and RE optimization. The exhibit is a portfolio composition map of the MV and RE optimal asset allocations in Figure 2; it displays the allocations from minimum risk on the left-hand side of the charts, to maximum return on the right-hand side of the charts. Each color represents a particular stock. A vertical slice of the chart demonstrates the weights of each stock in the portfolio at that level of risk. The left-hand panel is the composition map for MV efficiency; the right-hand panel represents RE optimality.

In the classical case no more than seven of the twenty stocks are included in the MV efficient frontier for the upper half of the curve, while three stocks are missing over the entire spectrum of risk. Similar examples can be found on nearly every MV frontier.

The composition map for the RE illustrates very different properties. RE optimality includes all assets. There is a smooth transition from one risk level to another. RE optimization is robust and fundamentally different in character and allocations

even when the two frontiers are similar in mean–variance space.

One of the most attractive features of RE optimized portfolios is that they often reflect investment sense. As one illustration, consider the maximum return MV and RE efficient portfolios in Figure 2. As Figure 3 indicates, the maximum return MV optimal portfolio represents a 100% bet in NTP stock. In contrast, the maximum return RE portfolio is very well diversified and a much less risky more acceptable investment.<sup>20</sup> More generally, RE optimized portfolios are often consistent with the intuition of professional managers across the entire frontier without the need for *ad hoc* constraints.

## 6 Simulation performance tests

RE optimized portfolios have many desirable investment properties relative to MV efficiency. However, the most important feature of RE optimization is its provable performance superiority on average. We use a simulation test rather than a back test to demonstrate the performance of the alternative

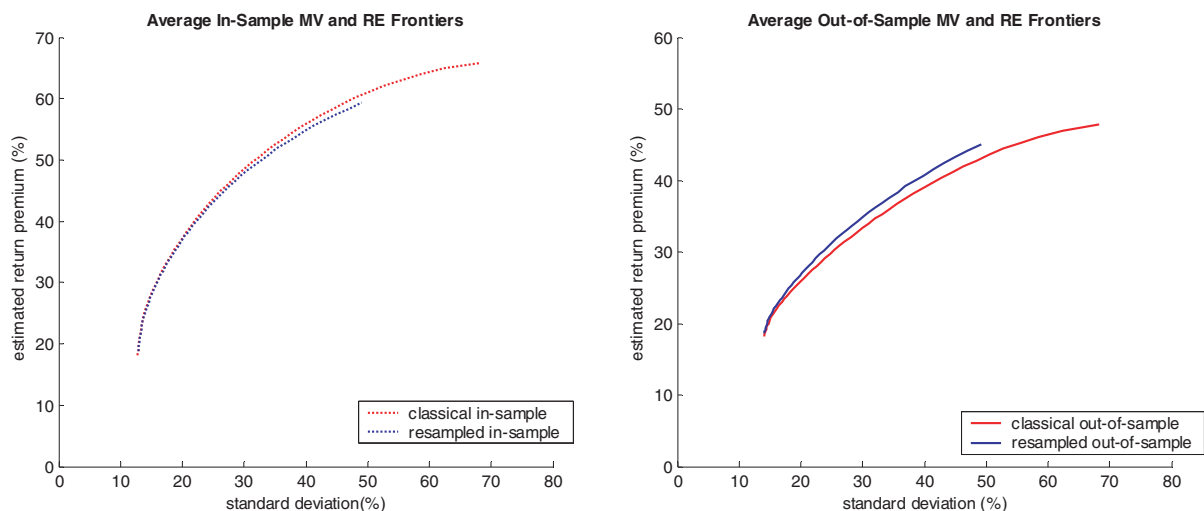
strategies, since a back test is time period dependent and there are not enough observations available to test for statistical significance.<sup>21</sup>

Following Jobson and Korkie (1981) and Michaud (1998, Ch. 6), we perform a simulation test to compare RE vs. MV optimization. In a simulation test, a referee is assumed to know the true values of asset risks and returns. The 20 stock risk-return data, shown in the Appendix, serves as the “truth” in our simulation experiments. The referee creates a simulated history and provides returns that are statistically consistent with the true risk-return estimates. These returns can either be thought of as historical observations of a stationary return distribution, or a number of noisy estimates of next period’s return. The Markowitz and RE investors compute their efficient portfolios based on the referee’s supplied returns. The referee uses the true risk-return values to score the optimized portfolios. Figure 4 gives the average of the results after many simulation tests.

The curves displayed in Figure 4 represent the averaged results from the simulation test. The left-hand panel displays the average MV and RE efficient

frontiers computed from the referee’s returns, the portfolios that were submitted to the referee for scoring. The higher (red) dotted curve is the MV efficient frontier; the lower (blue) dotted curve is the RE. The left-hand panel represents what the Markowitz and RE investors see on average given the referee’s data. The right-hand panel of Figure 4 illustrates the average results of how the submitted efficient frontier portfolios performed when the referee applied the true risk-returns. The higher (blue) solid curve represents the RE optimizer results; the lower (red) solid curve the Markowitz optimizer results. The right-hand panel of Figure 4 shows that the RE optimizer, on average, achieves roughly the same return with less risk, or alternatively more return with the same level of risk, relative to the Markowitz optimizer.<sup>22</sup>

The simulation experiment illustrates that the RE optimized portfolios are, on average, more effective at improving risk-adjusted investment performance.<sup>23</sup> RE optimized portfolios perform better because they are better risk managed and avoid the unrealistic use of investment information that characterizes Markowitz MV optimization solutions.<sup>24</sup>



**Figure 4** Comparison of MV and RE performance.

## 7 Optimizer or inputs

Asset managers have largely ignored the limitations of MV optimization technology in their investment process. Instead, the limitations of optimized portfolios in investment practice are typically addressed with proposals for improving risk-return estimates and allocating a considerable percentage of resources to the effort. Practitioner as well as academic financial literature is replete with various exotic statistical and investment proposals for improving optimization inputs. Our simulation studies suggest there is an additional route for improved optimized portfolio performance: a better optimizer. The notion that an improved optimizer may be as or more important than improved inputs is foreign to many.

Bayesian parameter estimation methods are among the most important tools in modern statistics for improving risk-return estimates.<sup>25</sup> Bayesian estimation in a simulation study often reflects a superior level of risk-return estimation than can be achieved in investment practice. Markowitz and Usmen (2003) address the issue of the relative importance of Bayesian estimation versus RE optimization. They develop a Bayesian diffuse prior procedure to enhance risk-return estimation. Using a simulation test framework as in Michaud (1998, Ch. 6), they compare the performance of their Bayesian enhanced risk-return estimates and MV optimization to RE optimization with unenhanced risk-return estimates. To their surprise, the RE optimized portfolios exhibited superior performance on average and in every one of their 30 individual tests.<sup>26</sup>

## 8 Markowitz–Usmen implications

The Markowitz–Usmen results may appear to defy investment intuition. One simple explanation is that MV optimization always assumes unrealistic

accuracy regardless of the quality of the inputs. The level of investment information presumed by the optimization process (16 decimal places of accuracy) is almost always inconsistent with investment information. The end result is that classical optimization overuses investment information and creates extreme portfolios that typically perform poorly in practice.

Bayesian parameter estimation is not an always better procedure; a perverse prior may lead to poorer, not better, estimates. Markowitz and Usmen (2003) cleverly finesse the perverse Bayesian prior problem by using a diffuse prior in their study.<sup>27</sup> The perverse Bayesian prior is a major concern in statistical estimation theory. Efron (2005), in his American Statistical Association presidential address, advocates “empirical Bayes” methods to avoid the effects of a perverse prior in Bayesian analysis while attempting to retain its improvement of statistical estimation power. The resampling and bootstrap methods Efron advocates conceptually resemble RE optimization.

In the RE optimization process, economically meaningful portfolio constraints serve as the fundamental mechanism for improving RE optimized portfolios. For example, nonnegativity constraints express the diffuse Bayesian belief that all assets in the optimization universe are desirable investments.<sup>28</sup> Diffuse Bayesian constraint priors operate on RE optimized portfolio structure by imposing reliable constraints on portfolio weights for each resampled optimized portfolio. The improvement in RE optimized portfolios results from the interaction of Bayesian prior constraints with the RE optimization averaging process. The Markowitz–Usmen results can be rationalized by observing that nonnegative constraints and RE optimization may often be more restrictive in defining optimality than Bayesian parameter estimation in classical optimization.<sup>29</sup>

While it is interesting to compare the relative value of statistically improved inputs versus RE optimization, the procedures are conceptually complementary, not exclusive. Procedures for improving the investment value of estimates are always worthwhile.

## 9 Certainty level and RE optimality

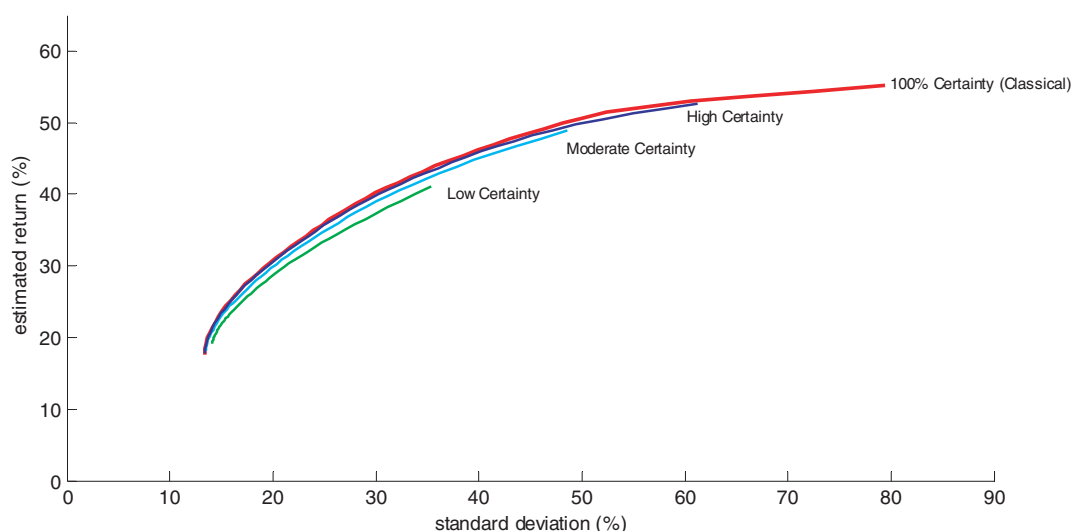
Up until now, we have avoided the issue of level of information in risk-return estimates. Except as noted, our performance studies were based on simulating 10 years of monthly returns. While consistent with our historical data set and appropriate from an academic point of view, the simulation test framework has limitations for practice.

Very generally, investors do not know that their risk-return estimates reflect the information contained in 10 years of stationary monthly returns as reflected in simulations of the Appendix data. One reason is that risk-return estimates in practice are rarely based solely on historical return data. More importantly, certainty varies by investor, strategy, and economic and market outlook among many factors. To be useful, RE optimization must be

customizable to the perceptions of many kinds of investors, strategies, and investment horizons.

The number of simulated observations used to compute the simulated MV efficient frontiers in each resampling of risk-return estimates is a free parameter of the RE optimization process. As the number of observations becomes large, every set of simulated risk-return estimates becomes increasingly similar to the original set, and the REF approaches the MV efficient frontier. When the number of observations becomes small, the REF approaches the no-information prior efficient portfolio.

The number of simulated observations is a natural way to model the amount of confidence an investor has in their risk-return estimates.<sup>30</sup> Figure 5 provides an illustration of the resulting frontiers at different levels of information. As confidence increases, REF portfolios become less diversified because they use information more actively as the frontier approaches the classical. The number of simulated observations is a mechanism for tuning the optimization process according to the level of certainty and time horizon associated with estimates.<sup>31</sup>



**Figure 5** Classical and Resampled Efficient Frontiers: Forecast Confidence levels.

Considerations of information confidence level lead to an important conclusion. RE optimization is simply a generalization of Markowitz MV optimization that allows the investor to control the amount of confidence they have in their investment information in the optimization process.

## 10 Optimality and investability

Theoretically the REF includes nonzero allocations for all the assets in the optimization universe. Given enough simulations, there will be at least one that includes a nonzero allocation to a given asset in a simulated MV frontier. This is similar to the market portfolio or global minimum variance portfolio, both of which theoretically include an allocation to every asset in the universe. In contrast, MV optimization typically excludes many assets in the efficient frontier. This important difference highlights the proper role of an optimizer: optimally weighting assets in the optimization universe. Choosing investable assets is the role of the analyst, not the optimizer. Excluding assets is a serious, though often ignored, negative property of classical linear constrained MV optimization.

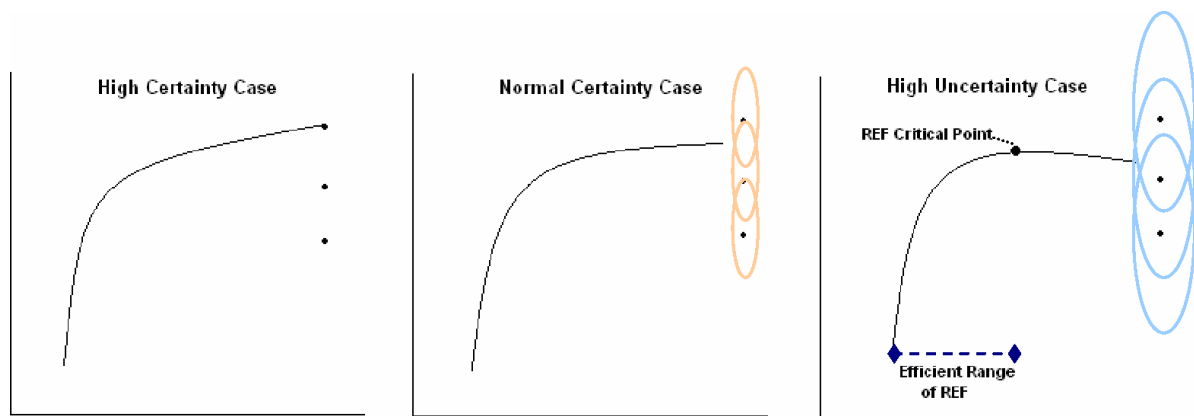
From a practical point of view, however, many REF allocations may be investment nuisances. The issue

of “too small” allocations is one of investability, not optimality. Many nonzero REF optimal allocations may not be investable depending on the amount of invested capital and other liquidity considerations.<sup>32</sup> Michaud and Michaud (2003) teach that the proper procedure for dealing with investability in the context of RE optimality is to impose investability constraints during a second optimization step using mixed integer programming best approximation. This two step process permits the investor to overlay the optimal portfolio with practical trading and other considerations.

## 11 The REF maximum return point

A statistical approach to portfolio optimality leads to some significant differences from classical MV optimization. One significant difference is that the REF curve may peak and then slope downward.<sup>33</sup> If the maximum risk portfolio on the REF is not also the maximum return portfolio, we call this the “maximum return point” (MRP) of the REF. The possible existence of an MRP is an important and very useful property of REF portfolio optimality.

Figure 6 illustrates how the REF MRP may arise. In each panel there are three high risk assets; uncertainty is indicated by the elliptical confidence region



**Figure 6** Forecast Confidence levels and the REF maximum return point.

*Source:* Michaud and Michaud (2008a, Chap. 6).

around each point. The left-hand panel depicts the Markowitz case, where perfect certainty in information implies the risk and return are point estimates. Here, the maximum return portfolio includes only the maximum return asset and no MRP is possible. The middle panel exhibits a moderate amount of uncertainty in the return distribution of the three assets. Since the true maximum return asset can no longer be known with certainty, the REF maximum return portfolio includes a significant amount of the middle asset (although not as much as the top asset) and has lower expected return than the Markowitz maximum return portfolio. The right-hand panel depicts a high level of uncertainty in the expected return estimates of the three assets. Since there is little statistical distinction between the three assets, the REF includes significant allocations in all three assets. In this case an MRP may emerge where the REF has a downward-sloping inefficient segment. Any risk beyond the MRP is not optimal and not on the REF by definition.

The MRP arises because RE optimization uses information from all assets in the optimization universe to form portfolio return expectations—the presence of risky low return assets means similarly risky high return assets may not realize such high returns in the future. The MRP defines a new upper-bound on the level of risk which we can be confident will lead to increasing expected portfolio return.<sup>34,35</sup>

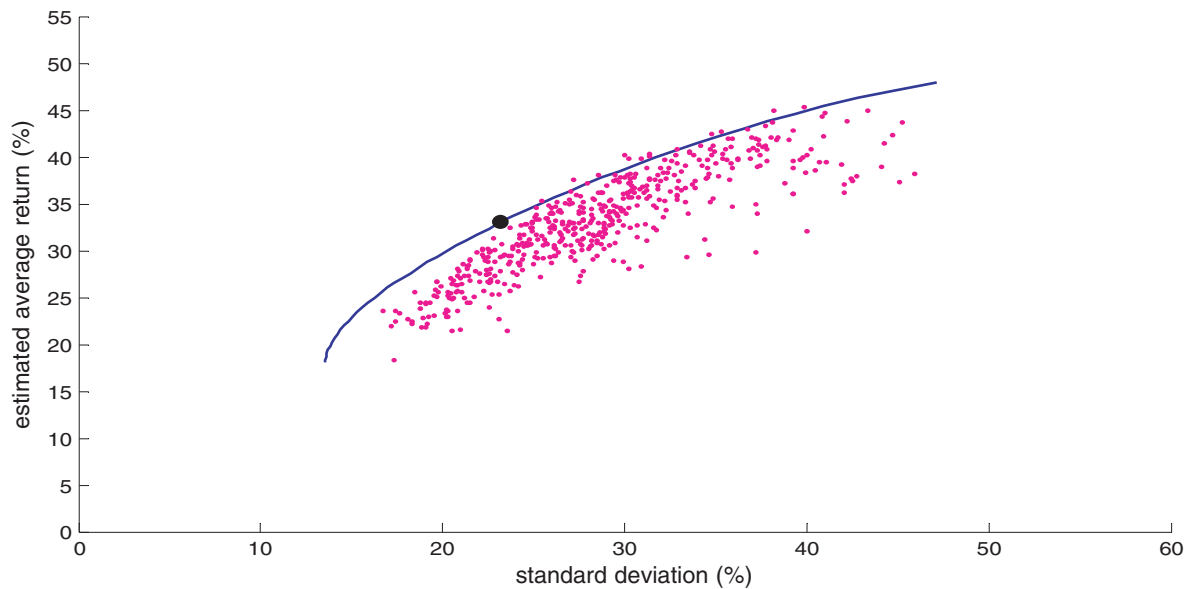
An REF MRP is relatively rare in asset allocation studies, because assets in the optimization universe often have similar attractive risk-return characteristics. In contrast, an REF MRP is often observed in large stock universe equity portfolio optimizations. This is because the optimization universe may have many assets that have little return and high risk. The identification of the MRP prevents the overuse of inferior investments as they generally have small allocations before the MRP, but may have large allocations at greater risk levels. The MRP helps

to identify the amount of information in forecasts, and has useful applications to the proper scaling of return forecasts.

But why would an investor include an inferior asset in the optimization universe in the first place when shorting is not allowed? Merton (1987) teaches that the optimization universe should always be defined in terms of what you know. An asset with high risk and low return may have no statistically significant investment information. An optimizer is not designed to tell which investments are to be ignored. This is much like including inappropriate data in a Bayesian prior. However, Merton's advice in the context of a large stock optimization universe that only includes statistically significant equities may result in unacceptable tracking error risk with many underrepresented sectors and industries. One simple solution for resolving these competing objectives is described in Michaud and Michaud (2005a): include an index-weighted composite asset of the statistically insignificant stocks in the index of the optimization.<sup>36</sup> The Merton principle of investing only in what you know remains the appropriate one.

## 12 RE rebalancing

Portfolio optimization information is often insignificant. In particular, many estimates of stock alphas in an equity portfolio optimization relative to a large index are likely to be statistically insignificant. MV optimization is insensitive to investment insignificance resulting in frequent portfolio rebalancings that may have no investment value. Familiarity with the limitations of MV optimization for asset management has led to numerous *ad hoc* rebalancing rules. Managers often rebalance on an arbitrary time schedule, such as monthly, quarterly, or annually. Alternatively, portfolio weights are assigned arbitrary ranges, such as plus or minus 10%; if an asset weight in the current portfolio is out of the range, the investor rebalances



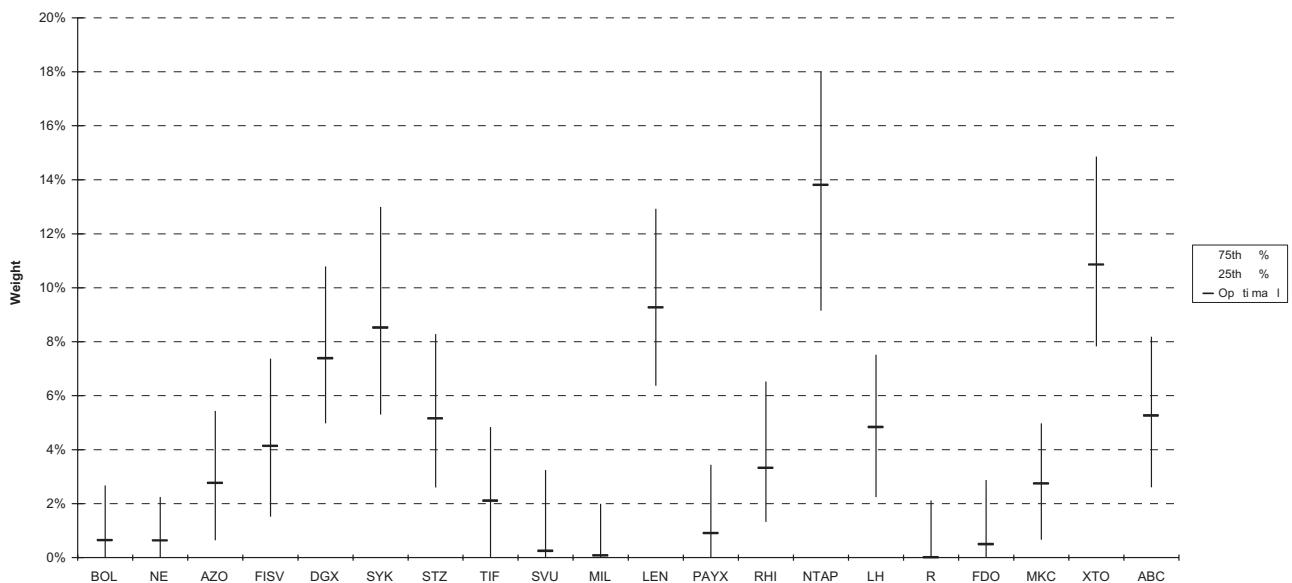
**Figure 7** Resampled Efficiency and associated efficient portfolios.

the portfolio. Portfolio rebalancing in practice is essentially portfolio maintenance with little, if any, evidence of investment effectiveness. Conceptually, portfolio rebalancing should be based on statistically significant differences relative to optimality.<sup>37</sup> RE rebalancing provides the appropriate tool for understanding statistically significant differences.

Figure 7 illustrates how the RE rebalancing rule works. By definition, every portfolio on the REF is an average of properly associated portfolios. The large dot indicates a selected “middle” optimal REF portfolio.<sup>38</sup> The small dots (magenta) surrounding the selected portfolio indicates the associated simulated efficient frontier portfolios. The associated portfolios provide a framework for defining “nearness” for a given metric, risk level, and confidence level. The relative variance or “tracking error” between the optimal and associated portfolios provides a convenient metric.<sup>39</sup> Using this metric, the relative variance is computed for the current portfolio and for each associated optimal portfolio relative to the indicated REF optimal

portfolio. The RE rebalancing rule defines the “need-to-trade” probability as the percent of associated portfolio relative variances closer to the optimal portfolio compared to the current portfolio. A 10% need-to-trade probability implies that only 10% of the associated portfolio variances are closer than the current portfolio in the metric.

The RE portfolio rebalancing rule has some interesting investment properties. The need-to-trade probability measures portfolio structure similarity by taking into account the means, variances, and covariances of all the assets in the optimization universe. An overweight in large cap stocks and an underweight in mid cap stocks are not as significant as a similar overweight in emerging markets and underweight in non-US bonds. The rebalance probability is a measure of how similar the current portfolio will perform relative to the optimal portfolio in the investment period. RE rebalancing provides a statistically rigorous and investment meaningful measure of when rebalancing is necessary.<sup>40</sup>



**Figure 8** Middle REF portfolio bounds (25th/75th percentile asset weight ranges).

The RE need-to-trade probability is a portfolio rebalancing rule. It does not indicate portfolio imbalances at the asset level. However, the associated portfolios in Figure 7 can also be used to compute a statistical analysis that is useful for understanding imbalances at the asset level and for portfolio monitoring. Each associated portfolio has an optimal allocation to each asset in the optimization universe. Percentile ranges and standard deviations can be computed for each asset, resulting in a linear regression type statistical analysis for the optimal portfolio.

Figure 8 displays the results of such a statistical analysis for the Appendix data for an REF middle optimal portfolio. There may be no direct relationship between the size of an allocation and the size of the bounds. For instance, note that the optimal allocation to TIF and MKC are nearly the same though different in significance or importance in the optimization. When the current portfolio weights lie outside the percentile bounds, rebalancing may be indicated. The procedure can also be used to enhance the Sharpe (1992)

returns-based style analysis procedure by including statistical estimates of the significance of the style coefficients.

The RE rebalancing rule has been enhanced with meta-resampling techniques since it was described in Michaud (1998, Ch. 7).<sup>41</sup> The new procedure provides substantially enhanced and relatively uniform statistical power across the entire range of portfolios on the REF. The meta-resampling rebalancing method features customization capabilities for various investment strategies, applications, and asset management characteristics. The enhanced procedure also markedly reduces the skewness of asset weight distributions.<sup>42</sup>

### 13 Benchmark-relative optimization

Benchmarks arise naturally in many asset management contexts. Active manager performance is typically evaluated relative to the residual return or tracking error with respect to some benchmark or index.<sup>43</sup> MV optimization is defined as maximizing

portfolio residual return relative to a given level of residual or tracking error risk.<sup>44</sup> Estimation error uncertainty is as important in benchmark- or index-relative optimization as in sign-constrained MV optimization. However, RE optimization requires accommodation to the presence of (relatively) unattractive, as well as attractive, assets in the optimization universe.

Non-*ad hoc* optimization constraints should be an expression of your beliefs. They act as Bayesian priors in MV as well as RE optimization.<sup>45</sup> Index-relative sign-consistent constraints (less/greater than or equal to the index weight for negative/positive residual return assets) perform the same Bayesian function as sign constraints in total return optimizations.<sup>46</sup> Also, as in sign-constrained total return optimization, simulation tests show that RE index-relative optimization with appropriate index-relative constraints outperforms MV optimization.<sup>47</sup>

Index-relative optimization can be described as long-short optimization without leverage. Equity portfolio long-short optimization is currently very topical. This is because many standard equity portfolio indices, such as the S&P 500, are highly asymmetric. In long-only optimization, index weight asymmetry affects the nature of the optimized portfolio. This is because small index-weight stocks can only receive a small underweight relative to negative information but can be overweighted relative to positive information with little limitation. In contrast, large index-weight stocks are less biased by positive or negative information relative to index weights. Assuming estimation error is relatively index-weight symmetric, the optimizer may invest heavily in attractive small index-weight stocks but cannot offset this with large underweights in unattractive small index-weight stocks, often resulting in a small index-weight stock portfolio bias.<sup>48</sup>

Proper optimization design requires avoiding pitfalls such as a small cap bias for asymmetric indices. A long-short 120/20 or 130/30 strategy may allow relatively symmetric constraints on stocks in an optimization universe.<sup>49,50</sup> Two-fund long-short strategies may also limit the impact of index asymmetric constraints on portfolio optimization.<sup>51</sup> Another simple alternative is a two-stage optimization; the first stage determines the theoretically optimal portfolio with index-symmetric constraints; the second stage attempts to closely replicate this portfolio while satisfying investability constraints.

Constraints reflecting beliefs are intimately associated with the particular character of investment information for a given asset manager or strategy. RE optimization, properly implemented may be as important for improved long-short investing as for more traditional contexts. Understanding the non-*ad hoc* Bayesian role of constraints is essential in managing inadvertent biases as well as reaping the benefits of a statistical approach to portfolio optimization.<sup>52</sup>

#### 14 Additional properties

RE optimization can be understood as an information level constraint. For example, in a two asset case, the REF lies on, but does not extend as far as, the classical efficient frontier. The shorter REF indicates that the risk likely to be rewarded with increased return is less than the maximum return asset. It is noteworthy that this result is consistent with much professional practice; few recommended asset allocations involve portfolios near the top of the MV efficient frontier. RE optimization quantifies this important investment intuition. Even for two asset cases, RE optimization provides useful investment information.

The REF may exhibit puzzling behavior in in-sample MV space. One example is that the REF may have a non-concave segment. While a linear combination of portfolios on the REF may lie above the REF, this fact has no particular consequence since portfolios always exist above the REF in MV space. REF non-concavity may be associated with a gap or paucity of estimates in a region of the efficient frontier or estimates inconsistent with rational pricing.<sup>53</sup> Alternatively, non-concavity may simply indicate that the simulation has not converged and requires more simulations.

Since the REF is always below the classical frontier in MV space, in-sample utility analysis always trivially indicates less “utility” or “information ratios” for RE optimal portfolios. RE optimization does not maximize in-sample parameters, rather it accounts for the possibility these parameters are wrong.

## 15 Conclusion

RE technology is a generalization of Markowitz MV optimization that allows investors to include the level of certainty in risk-return estimates in the optimization, rebalancing, and monitoring process. Avoiding the literal use of investment information implied by Markowitz optimization is a necessary though not sufficient condition for improved risk-adjusted performance. By factoring in estimation error in a Bayesian-like framework, RE optimization avoids unreliable and self-defeating principles of design and management that follow from in-sample parameter certainty MV portfolio optimization analytics. RE optimization also benefits from improved Bayesian forecasting and other improvements to the optimization inputs. When properly used, a new framework emerges that provides a far more reliable and productive route for research and effective asset management.

## Appendix: Twenty stock data set

A Standard & Poor’s 500 list of stocks and their market capitalizations were taken as of October 2006. The market capitalizations were discounted back via monthly total returns to approximate market capitalizations in January 1997. The 20 Large Cap stock set was taken from a random sample of the 100 largest of these market capitalization stocks. Ten years of complete monthly returns for the sample spans January 1997 through December 2006.

Ticker	Asset name	Return	Standard deviation
BOL	Bausch & Lomb	13.5	36.9
NE	Noble Corporation	24.4	46.7
AZO	AutoZone Inc.	19.9	33.0
FISV	Fiserv Inc.	20.7	31.7
DGX	Quest Diagnostics	36.9	49.0
SYK	Stryker Corp.	25.8	33.5
STZ	Constellation Brands	32.4	54.2
TIF	Tiffany & Co.	23.9	41.6
SVU	SUPERVALU Inc.	17.2	29.9
MIL	Millipore Corp.	13.9	38.2
LEN	Lennar Corp.	32.0	38.6
PAYX	Paychex Inc.	19.9	32.3
RHI	Robert Half International	19.9	40.3
NTAP	Network Appliance	55.2	79.4
LH	Laboratory Corp. of America Holdings	37.8	56.0
R	Ryder System	13.1	29.3
FDO	Family Dollar Stores	21.2	31.3
MKC	McCormick & Co.	15.9	19.3
XTO	XTO Energy Inc.	47.5	58.4
ABC	Amerisourcebergen Corp.	21.1	38.9

	BOL	NE	AZO	FISV	DGX	SYK	STZ	TIF	SVU	MIL	LEN	PAYX	RHI	NTAP	LH	R	FDO	MKC	XTO	ABC
BOL	1	0	0.08	0.17	0.06	0.23	0.04	0.21	0.25	0.12	0.03	0.07	0.13	0.08	0.08	0.22	0.17	0.15	0	0.24
NE	0	1	0.08	0.21	0.03	0.21	0.05	0.25	-0.09	0.25	0.27	0.15	0.07	0.47	0.21	0.14	0.21	0.06	0.43	-0.05
AZO	0.08	0.08	1	0.1	0	0.27	0.09	0.23	0.32	0.14	0.34	0.08	0.03	0.06	0.03	0.19	0.35	0.14	0.05	0.12
FISV	0.17	0.21	0.1	1	0.09	0.29	-0.05	0.35	0.35	-0.01	0.24	0.26	0.23	0.11	0.02	0.24	0.13	0.13	0.01	0.17
DGX	0.06	0.03	0	0.09	1	0.04	0.59	0.04	0.1	0.06	-0.01	0.15	0.09	0	0.71	0.02	0.03	0.05	0.39	0.22
SYK	0.23	0.21	0.27	0.29	0.04	1	0	0.15	0.26	0.2	0.28	0.13	0.09	0	0.08	0.12	0.27	0.19	0.15	0.13
STZ	0.04	0.05	0.09	-0.05	0.59	0	1	0.03	0.21	-0.01	0.01	0.11	-0.04	-0.07	0.48	0.09	0.24	0.16	0.38	0.09
TIF	0.21	0.25	0.23	0.35	0.04	0.15	0.03	1	0.2	0.21	0.3	0.25	0.22	0.41	-0.03	0.33	0.25	0.14	0.11	-0.08
SVU	0.25	-0.09	0.32	0.35	0.1	0.26	0.21	0.2	1	0.13	0.24	0.29	0.32	0	0.18	0.24	0.33	0.34	0.04	0.21
MIL	0.12	0.25	0.14	-0.01	0.06	0.2	-0.01	0.21	0.13	1	0.07	-0.11	0.13	0.31	0.09	0.06	-0.01	-0.01	0.14	0.03
LEN	0.03	0.27	0.34	0.24	-0.01	0.28	0.01	0.3	0.24	0.07	1	0.24	0.09	0.18	0.03	0.19	0.46	0.25	0.16	0.24
PAYX	0.07	0.15	0.08	0.26	0.15	0.13	0.11	0.25	0.29	-0.11	0.24	1	0.29	0.35	0.12	0.12	0.34	0.18	0.18	0.02
RHI	0.13	0.07	0.03	0.23	0.09	0.09	-0.04	0.22	0.32	0.13	0.09	0.29	1	0.28	-0.02	0.29	0.09	0.02	-0.1	0.18
NTAP	0.08	0.47	0.06	0.11	0	0	-0.07	0.41	0	0.31	0.18	0.35	0.28	1	0.01	0.07	0.12	-0.09	0.12	-0.09
LH	0.08	0.21	0.03	0.02	0.71	0.08	0.48	-0.03	0.18	0.09	0.03	0.12	-0.02	0.01	1	0.06	0.09	0.09	0.41	0.19
R	0.22	0.14	0.19	0.24	0.02	0.12	0.09	0.33	0.24	0.06	0.19	0.12	0.29	0.07	0.06	1	0.29	0.12	0	0.05
FDO	0.17	0.21	0.35	0.13	0.03	0.27	0.24	0.25	0.33	-0.01	0.46	0.34	0.09	0.12	0.09	0.29	1	0.34	0.24	0.13
MKC	0.15	0.06	0.14	0.13	0.05	0.19	0.16	0.14	0.34	-0.01	0.25	0.18	0.02	-0.09	0.09	0.12	0.34	1	0.29	-0.02
XTO	0	0.43	0.05	0.01	0.39	0.15	0.38	0.11	0.04	0.14	0.16	0.18	-0.1	0.12	0.41	0	0.24	0.29	1	0.03
ABC	0.24	-0.05	0.12	0.17	0.22	0.13	0.09	-0.08	0.21	0.03	0.24	0.02	0.18	-0.09	0.19	0.05	0.13	-0.02	0.03	1

*Twenty stock data set correlations*

## Notes

- <sup>1</sup> Jobson and Korkie (1981). In a more recent study, DeMiguel *et al.* (2008) test the performance of 14 models of unbounded MV portfolio choice in the context of estimation error and find, as in Jobson and Korkie, that none seem reliable improvements over equal weighting.
- <sup>2</sup> RE optimization, invented by Richard Michaud and Robert Michaud, is protected by United States and Israeli patents and patents pending worldwide. New Frontier Advisors, LLC (NFA) is exclusive worldwide licensee.
- <sup>3</sup> See Michaud and Michaud (2008a) for further discussion.
- <sup>4</sup> See Michaud and Michaud (2005b) for more discussion.
- <sup>5</sup> An early reference is Bawa *et al.* (1979). More recent examples include Jorion (1986), Harvey *et al.* (2006), DeMiguel *et al.* (2008) and Brandt (2007).
- <sup>6</sup> See also Cremers *et al.* (2004).
- <sup>7</sup> See Boyd and Vandenberghe (2004) for an up-to-date review of algorithms for solving convex optimization problems including Markowitz portfolio optimization.
- <sup>8</sup> Michaud (1998, Ch. 3) and Michaud and Michaud (2008a, Ch. 3) review traditional critiques of MV optimization.
- <sup>9</sup> The IEEE 754 digital computer standard implies data storage of essentially 10 single or 16 double precision decimals of floating point data. The authors use double precision software for computation.
- <sup>10</sup> The problem of unrealistic accuracy representation of information and unstable or knife-edge optimizations is a general, though widely ignored, problem affecting a variety of industrial, military, medical, robotics, economics, as well as financial applications. The methods described in this paper may also apply to these fields.
- <sup>11</sup> This is typically done with resampling, or bootstrapping. In this paper, bootstrapping generally refers to the technique of redrawing historical observations with replacement. Resampling typically refers to recreating the simulated history from an assumed probability distribution such as multivariate normality. Non-normal resampling may be appropriate in some applications but is beyond the scope of this report. See Efron and Tibshirani (1993) for an authoritative discussion of these procedures. In practice resampling is typically convenient since few investment strategies are based solely on risk-return estimates from historical return data.
- <sup>12</sup> Following Jobson and Korkie (1981), the optimizations are illustrated based on the risk and returns of 20 US stocks randomly chosen from 100 largest capitalization stocks in the S&P 500 index with continuous monthly returns from January 1997 through December 2006. The list of stocks, their annualized average returns, standard deviations and correlations over the period and further details are given in the Appendix.
- <sup>13</sup> Each simulated efficient frontier is defined to be consistent with the uncertainty in the original data set, i.e., 120 monthly simulated returns are computed for each asset; the simulated returns assume a multivariate normal distribution. The optimized portfolios are constrained to have nonnegative weights and sum to one. The simulated MV efficient frontier procedure may have a number of variations. For example, the return distribution assumption can be changed or historical data may be bootstrapped. In many cases in practice the number of simulated returns for computing the simulated MV efficient frontiers is not associated with a historical return data set and must be assumed. This issue leads to the important (patent pending) concept of Forecast Confidence™ (FC) level discussed further below and in Michaud and Michaud (2004a).
- <sup>14</sup> In practical applications, many more efficient frontiers are simulated. However, it serves our pedagogical purposes in this case to display only 25 simulated frontiers.
- <sup>15</sup> The simulated MV efficient frontiers that are the starting point of RE optimization are consistent with Efron's (2005) "empirical Bayes" procedure. As in Efron, resampling is employed as a powerful statistical tool for understanding variability in parameter estimation and for creating "objective" priors.
- <sup>16</sup> The construction process described is consistent with the Appendix in Michaud (1998, Ch. 6). It highlights how rational agents may make investment decisions that lead to REF optimality. Markowitz and Usmen (2003) express concern that RE optimization may require revision of expected utility axioms. As our discussion shows, RE optimization is fundamentally based on expected utility considerations. Their concerns may be due to the original description of the procedure in the body of the Michaud (1998) text that used a heuristic construction process devoid of utility function considerations. We note that approximation algorithms of the REF have been found convenient and investment effective relative to compute-efficiency and statistical stability estimation considerations (see Michaud 1998, Ch. 6, Appendix). Our views on rationality axioms and rule based systems are discussed in Michaud (2003, fn. 6).
- <sup>17</sup> Resampling and bootstrap methods in statistics are generally concerned with exploring the variability implicit in

historical data as in Efron (2005). RE optimization uses the variability exposed by resampling to define a new statistic that did not exist before.

- <sup>18</sup> The risk range of the REF may be greater than MV efficiency in some cases; this issue is not material to our discussion here.
- <sup>19</sup> It is a necessary condition that the risk spectrum for estimation error sensitive MV portfolio efficiency converges to the no-information portfolio as uncertainty increases. This property contradicts the properties of the heuristic Feldman (2003) and Ceria–Stubbs (2005) methods where the risk spectrum is always constant and equal to classical efficiency whatever the level of certainty in investment information. Note also, in contrast to the teachings in Chopra and Ziemba (1993), Figure 2 shows that considering estimation error in risk as well as return is fundamentally important for defining portfolio optimality under information uncertainty.
- <sup>20</sup> Note each asset weight in the maximum return REF portfolio is equal to the probability that it is truly the maximum return asset.
- <sup>21</sup> A simulation proof of superiority requires assumptions about the “truth” distribution of assets for the results to be valid. In particular the truth data set and simulation framework has to make financial sense. A historical data set with negative average returns for some assets does not make sense as a truth in a simulation proof with sign constraints. In more subtle cases some assets may statistically dominate others over some spectrum of risk and return. While RE optimization can not be proven to have higher out-of-sample risk-return in all possible simulation tests, properly implemented it outperforms for financially relevant cases of practical interest. Michaud and Michaud (2008a, Ch. 6, Appendix B) provide a geometric proof of superiority that is data set independent. As discussed below, simulation tests for the index-relative and long–short case require certain additional considerations for validity (see also Michaud and Michaud 2008a, Ch. 9 for further discussion).
- <sup>22</sup> Markowitz and Usmen (2003) replicated the Michaud (1998, Ch. 6).
- <sup>23</sup> The results assume a MV optimization for total or real return, sign and budget constrained, optimized portfolios. The results can be generalized to include leverage. The index- or benchmark-relative, and long–short optimization frameworks are discussed in a subsequent section; simulation proofs are available in Michaud and Michaud (2008a, Ch. 9). There is no performance improvement for unbounded MV optimization.
- <sup>24</sup> Knight and Satchell (2006) find no benefit to RE optimization. However, they only examine the unbounded case.
- <sup>25</sup> A useful reference is Robert (1994).
- <sup>26</sup> The Harvey *et al.* (2008) tests disputing Markowitz and Usmen (2003) are based on a different procedure for computing RE optimized portfolios than that used in Markowitz and Usmen. This alternative procedure has been shown to have inferior out of sample performance to that used by Markowitz and Usmen (see Michaud 1998, Chapter 6, Appendix). This difference could explain their results. Also, their one-step-forward test is actually in- not out-of-sample and their characterization of long-term investor behavior for rationalizing their results does not conform to any in our experience. See Michaud and Michaud (2008b) for further discussion.
- <sup>27</sup> Harvey *et al.* (2006) claim to improve on the Markowitz–Usmen Bayesian estimation procedure by handling higher moments. Their results however are not in the Levy–Markowitz (1979) framework assumed in this paper or used by most practitioners. Moreover they note that their results are addressed to improved in-sample expected utility rather than out-of-sample risk-adjusted performance as in Markowitz–Usmen and Michaud (1998, Ch. 6). While we do not dispute their in-sample results, our focus is on out-of-sample investment properties.
- <sup>28</sup> As indicated below, the diffuse Bayesian prior constraints for the index-relative or long–short case are different.
- <sup>29</sup> Note that Harvey *et al.* (2006) characterize the Michaud optimization resampling approach as changing the order of Bayesian integration. The Markowitz–Usmen results indicate that order of integration is non-trivially important. The order of integration is important because diffuse constraints may be highly reliable informative Bayesian priors on optimal portfolio structure.
- <sup>30</sup> To facilitate the user experience we have created a Forecast Confidence™ (FC) level scale ranging from 1 to 10, indicating very low to very high information level. On this scale Markowitz optimization is an 11 and complete uncertainty 0. See further discussion in Michaud and Michaud (2004a).
- <sup>31</sup> See Michaud and Michaud (2004a) for applications.
- <sup>32</sup> See also Michaud (1998, Ch. 12, p. 135).
- <sup>33</sup> See Michaud and Michaud (2004b).
- <sup>34</sup> Preliminary simulation tests are consistent with out-of-sample replication of in-sample REF MRPs.
- <sup>35</sup> The existence of the MRP is a useful way of justifying much institutional investment practice where assumed

tracking error in equity portfolio optimization is often far less than the maximum available. Alternatively, not knowing the limits of efficient risk in an optimized portfolio may adversely afflict asset management for many leveraged hedge fund managers.

- <sup>36</sup> Further description and implications of using the composite asset are given in the reference.
- <sup>37</sup> Academic methods for understanding statistically significant differences between portfolios, usually associated with tests of the Capital Asset Pricing Model, such as Shanken (1985), are based on an unbounded MV optimization framework and not useful for portfolios in practice. See also Michaud (1998, Ch. 7, App. A).
- <sup>38</sup> The middle portfolio used in Figures 7 and 8 is defined as the REF rank 31 portfolio from the spectrum of ranks 1 to 51 computed portfolios. The magenta simulated portfolios are all similarly ranked meta-resampled efficient frontier portfolios. See Michaud and Michaud (2008a, Ch. 6) for further details.
- <sup>39</sup> Other metrics may also be of interest depending on applications and can be used to define the RE rebalancing rule.
- <sup>40</sup> The portfolio rebalancing test can also be used to estimate a cost minimization statistically equivalent optimal portfolio.
- <sup>41</sup> Michaud and Michaud (2002). The new procedure was allowed as a patent in August 2005, Robert Michaud and Richard Michaud co-inventors. For the purposes of statistical estimation, every simulated MV efficient frontier becomes the basis of computing an associated Resampled Efficiency Frontier; the meta-resampling uses the portfolios on the associated Resampled Efficient Frontiers in the trading rule and estimates of statistical parameters. See Michaud and Michaud (2008a, Ch. 7) for additional details.
- <sup>42</sup> The Britten-Jones (1999) range estimates of optimized portfolio weights are only useful for unbounded MV optimization. Our meta-resampling procedure produces investment relevant coefficients for the practical case of linear constrained MV optimized portfolios with less variance.
- <sup>43</sup> The index or benchmark in asset allocation studies may be an index or a return liability. In equity portfolio optimization, portfolio risk is typically defined in a CAPM or APT framework. See Michaud and Michaud (2008a, Chs. 9 and 10) for further discussion.
- <sup>44</sup> The benchmark redefines risk in terms of the return of an investment relevant objective.

- <sup>45</sup> As Theil (1971, pp. 346–350) notes, analysts often have valuable extraneous information including introspection on the value of coefficients where “pure estimation” or dependence solely on the data is suboptimal. RE optimization methods may be viewed as an enabling technology for including reliable linear constraints in many multivariate estimation and maximization contexts.
- <sup>46</sup> There are important exceptions to simple rules of thumb that impose residual return index-weight relative sign constraints. One alternative, to define Bayesian prior constraints that depend on the statistical significance of the index-relative information, is beyond the scope of this report. These issues are addressed further in Michaud and Michaud (2008a, Ch. 9).
- <sup>47</sup> See Michaud and Michaud (2008a, Ch. 9).
- <sup>48</sup> A manager may consider a small cap portfolio bias desirable. The point of the example is to note that a small cap bias emerges in index-relative portfolio optimization from long-only constraints in asymmetric indices independent of any capitalization based information asymmetry.
- <sup>49</sup> Long–short optimized portfolios generally need to satisfy certain conditions (Jacobs *et al.*, 2006a) in order to have practical investment value. Some additional conditions may be necessary to control unlikely extreme events in the resampling process.
- <sup>50</sup> See Jacobs and Levy (2006b).
- <sup>51</sup> In the classical long–short case analyzed in Michaud (1993), before shorting, both the long and short portfolios are sign constrained creating relatively symmetric constraints in the total portfolio.
- <sup>52</sup> The discussion teaches that thoughtful constraints on optimal portfolio structure are your friends and should always be included. Note how avoiding estimation error in the unbounded MV optimization analytical framework in Clarke *et al.* (2002, 2006) leads to very different conclusions.
- <sup>53</sup> One important source of the inhomogeneous optimization universe problem is the disconnect between return and risk estimation in equity portfolio optimization. Alpha is typically estimated independently from commercial risk models. As a consequence, the return to risk distribution may not exhibit rational pricing.

## References

- Bawa, V., Brown, S., and Klein, R. (1979). *Estimation Risk and Optimal Portfolio Choice*. Amsterdam: North Holland.

- Black, F. and Litterman, R. (1992). "Global Portfolio Optimization." *Financial Analysts Journal* 48(5), 28–43.
- Brandt, M.W. (2007). "Portfolio Choice Problems." In Ait-Sahalia, Y. and Hansen, L.P. (eds.), *Handbook of Financial Econometrics*. Amsterdam: Elsevier Science Ltd.
- Britten-Jones, M. (1999). "Sampling Error in Mean-Variance Efficient Portfolio Weights." *Journal of Finance* 54(2), 655–671.
- Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge: Cambridge University Press.
- Campbell, J. and Viceira, L. (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. New York: Oxford University Press.
- Ceria, S. and Stubbs, R. (2005). "Incorporating Estimation Error into Portfolio Selection: Robust Efficient Frontiers." Axioma Working Paper.
- Chopra, V. and Ziemba, W. (1993). "The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice." *Journal of Portfolio Management* 19(1), 6–11.
- Clarke, R., deSilva, H., and Thorley, S. (2002). "Portfolio Constraints and the Fundamental Law of Active Management." *Financial Analysts Journal* 58(3), 48–66.
- Clarke, R., deSilva, H., and Thorley, S. (2006). "The Fundamental Law of Active Portfolio Management." *Journal of Investment Management* 4(3), 54–72.
- Cremers, J.H., Kritzman, M., and Page, S. (2004). "Portfolio Formation with Higher Moments and Plausible Utility." Revere Street Working Paper Series: Financial Economics (November).
- DeMiguel, V., Garlappi, L., and Uppal, R. (2008). "Optimal Versus Naïve Diversification: How Inefficient is the 1/N Portfolio Strategy?" *Review of Financial Services*. Forthcoming.
- Efron, B. (2005). "Bayesians, Frequentists, and Scientists." *Journal of the American Statistical Association* 100(469), 1–5.
- Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap*. New York: Chapman and Hall.
- Feldman, B. (2003). "Ibbotson's Approach to Resampling." Presented to the Ibbotson Client Conference (Fall) in Orlando.
- Frost, P. and Savarino, J. (1986). "An Empirical Bayes Approach to Efficient Portfolio Selection." *Journal of Financial and Quantitative Analysis* 21(3), 293–305.
- Frost, P. and Savarino, J. (1988). "For Better Performance Constrain Portfolio Weights." *Journal of Portfolio Management* 15(1), 29–34.
- Grinold, R. (1989). "The Fundamental Law of Active Management." *Journal of Portfolio Management* 15(3), 30–37.
- Grinold, R. and Kahn, R. (1994). *Active Portfolio Management*. Chicago: Irwin.
- Harvey, C., Liechty, J., Liechty, M., and Müller, P. (2006). "Portfolio Selection with Higher Moments." Duke University, Working Paper (October 16).
- Harvey, C., Liechty, J., Liechty, M., and Müller, P. (2008). "Bayes vs. Resampling: A Rematch." *Journal of Investment Management*, forthcoming.
- Jacobs, B., Levy, K., and Markowitz, H. (2006a). "Trimability and Fast Optimization of Long-Short Portfolios." *Financial Analysts Journal* 62(2), 22–34.
- Jacobs, B. and Levy, K. (2006b). "Enhanced Active Equity Strategies." *Journal of Portfolio Management* 32(3), 45–55.
- Jobson, D. and Korkie, B. (1980). "Estimation for Markowitz Efficient Portfolios." *Journal of the American Statistical Association* 75(371), 544–554.
- Jobson, D. and Korkie, B. (1981). "Putting Markowitz Theory to Work." *Journal of Portfolio Management* 7(4), 70–74.
- Jorion, P. (1986). "Bayes-Stein Estimation for Portfolio Analysis." *Journal of Financial and Quantitative Analysis* 21(3), 279–292.
- Jorion, P. (1992). "Portfolio Optimization in Practice." *Financial Analysts Journal* 48(1), 68–74.
- Knight, J. and Satchell, S. (2006). "Exact Properties of Measures of Optimal Investment for Institutional Investors." Presented to: The Centre for Advanced Studies in Finance and The Institute for Quantitative Finance and Insurance, March.
- Ledoit, O. (1997). "Improved Estimation of the Covariance Matrix of Stock Returns with an Application to Portfolio Selection." Anderson Graduate School of Management at UCLA, Working paper (March).
- Levy, H. and Markowitz, H. (1979). "Approximating Expected Utility by a Function of Mean and Variance." *American Economic Review* 69(3), 308–317.
- Markowitz, H. (1956). "The Optimization of a Quadratic Function Subject to Linear Constraints." *Naval Research Logistics Quarterly* 3(1/2), 111–133.
- Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investments*. 2nd edn., New York: Wiley. (Cambridge, MA: Basil Blackwell, 1991).
- Markowitz, H. and Usmen, N. (2003). "Resampled Frontiers Versus Diffuse Bayes: An Experiment." *Journal of Investment Management* 1(4), 9–25.
- Merton, R. (1987). "Presidential Address: A Simple Model of Capital Market Equilibrium with Incomplete Information." *Journal of Finance* 42(3), 483–510.

- Michaud, R. (1989). "The Markowitz Optimization Enigma: Is Optimization Optimal?" *Financial Analysts Journal* 45(1), 31–42.
- Michaud, R. (1993). "Are Long-Short Equity Strategies Superior?" *Financial Analysts Journal* 49(6), 44–49.
- Michaud, R. (1998). *Efficient Asset Management*. New York: Harvard Business School Press. (Now published by Oxford University Press.)
- Michaud, R. (2003). "A Practical Framework for Portfolio Choice." *Journal of Investment Management*, 2nd Quarter. (Reprinted in *The World of Risk Management*. Gifford Fong, H. (ed.), New Jersey: World Scientific, 2006.)
- Michaud, R. and Michaud, R. (2002). "Resampled Portfolio Rebalancing and Monitoring." *New Frontier Advisors Newsletter* (4th quarter).
- Michaud, R. and Michaud, R. (2003). "Optimal and Investable Portfolios." *New Frontier Advisors Newsletter* (June).
- Michaud, R. and Michaud, R. (2004a). "Forecast Confidence Level and Portfolio Optimization." *New Frontier Advisors Newsletter* (July).
- Michaud, R. and Michaud, R. (2004b). "Equity Optimization Issues-I." *New Frontier Advisors Newsletter* (November).
- Michaud, R. and Michaud, R. (2005a). "Insignificant Alphas and Heterogeneous Errors." *New Frontier Advisors Newsletter* (March).
- Michaud, R. and Michaud, R. (2005b). "Fundamental Law of Mismanagement." *New Frontier Advisors Newsletter* (July).
- Michaud, R. and Michaud, R. (2008a). *Efficient Asset Management*. 2nd edn. Oxford University Press: New York.
- Michaud, R. and Michaud, R. (2008b). "Defense of Markowitz-Usmen." Working paper. New Frontier Advisors. <http://www.newfrontieradvisors.com/research/Articles/ResampledEfficiencyArticles.html>.
- Robert, C. (1994). *The Bayesian Choice: A Decision-Theoretic Motivation*. New York: Springer.
- Roll, R. (1979). "Testing a Portfolio of Ex Ante Mean-Variance Efficiency." *TIMS Studies in the Management Sciences* 11, 135–149.
- Rubenstein, M. (1973). "A Comparative Statistics Analysis of Risk Premiums." *Journal of Business* 46(4), 605–615.
- Shanken, J. (1985). "Multivariate Tests of the Zero-Beta CAPM." *Journal of Financial Economics* 14(3), 327–357.
- Sharpe, W. (1992). "Asset Allocation: Management Style and Performance Measurement." *Journal of Portfolio Management* 18(1), 7–19.
- Theil, H. (1971). *Principles of Econometrics*. New York: Wiley.

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